Fitting Frequency Measurements to the Dispersion Relation for a Doubly Periodic Chain of Resonators

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A program was written which least squares fit the family of measured resonant frequencies in a side coupled cavity structure to the dispersion relation for a doubly periodic chain of resonators. Two versions were developed, one runs on the Apple family of computers and the other on IBM, or DOS, compatible machines. The program was modeled after a similar one developed at Los Alamos, reference 5.

Theory

If the side coupled cavity string is assumed to be doubly periodic then all accelerating cells have identical resonant frequency as do all coupling cells. In addition, the coupling coefficients must also be uniform along the chain of resonators.

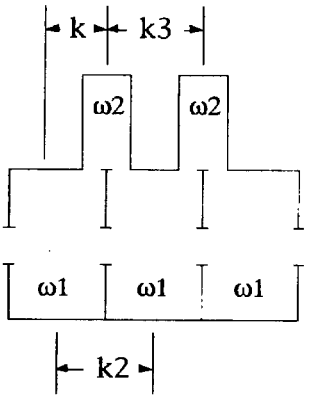


Figure 1. ω1 is the resonant frequency of an accelerating cell, ω2 for the side coupling cell, k is the coupling coefficient between the accelerating and side cells, k2 between accelerating, and k3 between side cells.

The dispersion relation between the resonant frequency, ω, of a mode and the phase shift between cells, ϕ, is given below, reference 1 and 2.

$$k^2 \cos^2\phi = (1 + k_2 \cos 2\phi - (\frac{\omega_1}{\omega})^2)\,(1 + k_3 \cos 2\phi - (\frac{\omega_2}{\omega})^2) \qquad \text{equation 1}$$

By multiplying out the coefficients, a quadratic in ω^2 can be formed and solved for ω

let
$$a = (1 + k_2 \cos 2\phi)(1 + k_3 \cos 2\phi) - k^2 \cos^2\phi$$
$$b = \omega_1^2 (1 + k_3 \cos 2\phi) + \omega_2^2 (1 + k_2 \cos 2\phi)$$
$$c = \omega_1^2 \omega_2^2$$

then
$$0 = a\omega^4 + b\omega^2 + c \qquad \text{equation 1a}$$

$$\omega = \sqrt{\frac{-b + q\sqrt{b^2 - 4ac}}{2a}} \qquad q = \begin{cases} -1 & \phi < \pi/2 \\ 1 & \phi > \pi/2 \\ -1 & \phi = \pi/2 \quad \& \quad \frac{\omega_1}{\sqrt{1-k_2}} < \frac{\omega_2}{\sqrt{1-k_3}} \\ 1 & else \end{cases}$$ equation 2

The limits on q allow the system to have positive real group velocity.

$$v_{group} = \frac{d\omega}{d\phi}$$

A stop band exists in the dispersion relation, equation 1, at $\phi=\pi/2$. The two possible solutions are shown below. The program, as written, selects the larger solution. In equation 1, w1, k2 and w2, k3 can be swapped in pairs while providing the same solution. The program swaps these pairs to maintain k3 larger than k2.

$$at\ \phi = \pi/2 \qquad \omega = larger\ of\ \frac{\omega_1}{\sqrt{1-k_2}}\ or\ \frac{\omega_2}{\sqrt{1-k_3}}$$

If the two end cells are half cells, formed by placing a ground plane at the center of the cavity perpendicular to the chain, then the lowest order standing wave mode occurs when the chain is π radians long. The highest order mode will have π radians of phase advance per cell. A chain of N cavities will have N resonance's with a phase advance between cells given by equation 3 below. Normally the end cells will be full cells which shifts the phase somewhat.

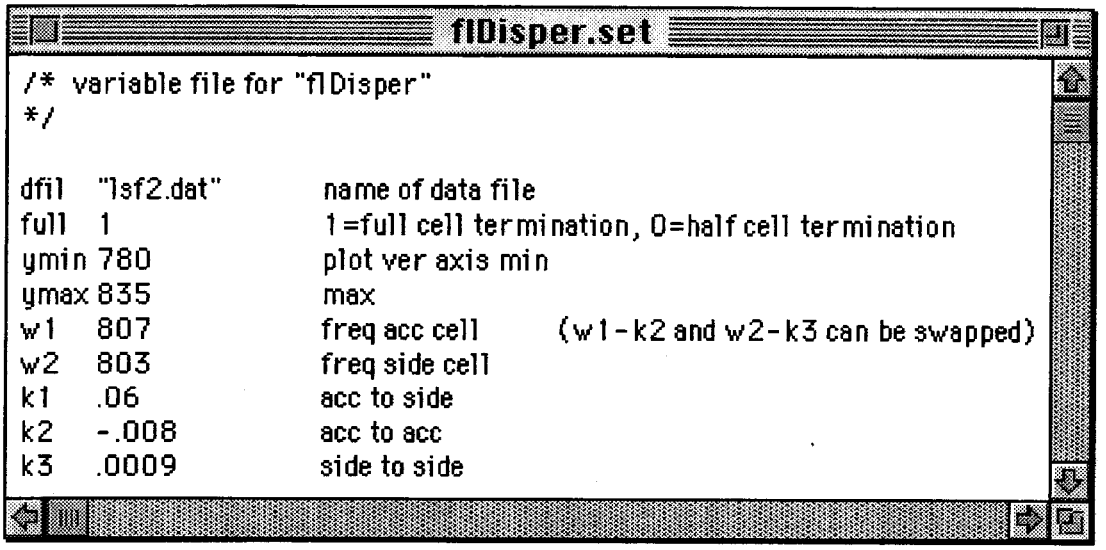
$$\phi = \begin{cases} \frac{i}{N+1} & half\ cell\ termination \\ \frac{i+1}{N+1} & full \end{cases} \qquad for \quad 0 \le i < N$$ equation 3

The $\phi=\pi/2$ mode has a voltage maximum at each accelerating cell and a voltage minimum π/2 radians away at each side cell. The adjacent accelerating cells are separated by π radians and thus have opposite polarity of accelerating field. The physical spacing of the accelerating cells is chosen so the field reverses as the beam travels from one cell to the next.

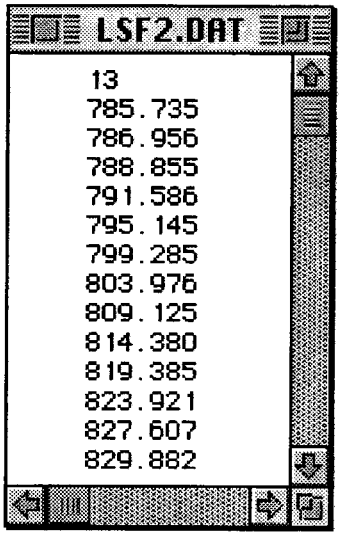
Program

The program uses the gradient expansion algorithm described in reference 3. The algorithm searches for a least squared difference between the data points and equation 2. The search is terminated when the change in chi squared becomes less than 1 part per million or 20 iterations whichever comes first.

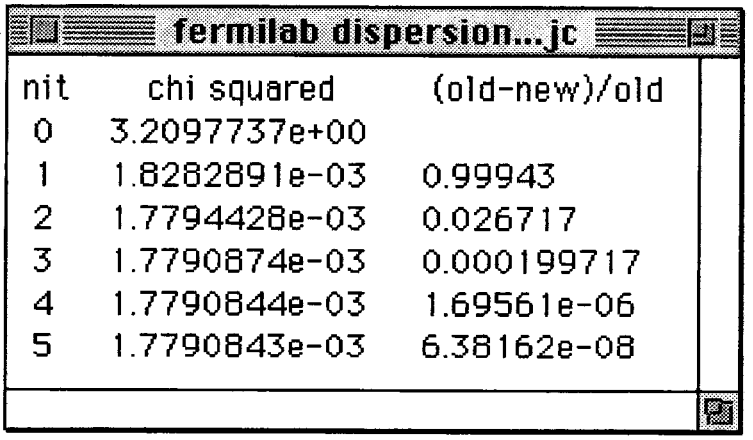
When the program is launched from the finder it reads the file 'flDisper.set'. As shown below it specifies the file containing the frequency measurements and allows the selection of full or half end cells as well as the plot vertical scale. The starting values for the gradient search are also entered here.

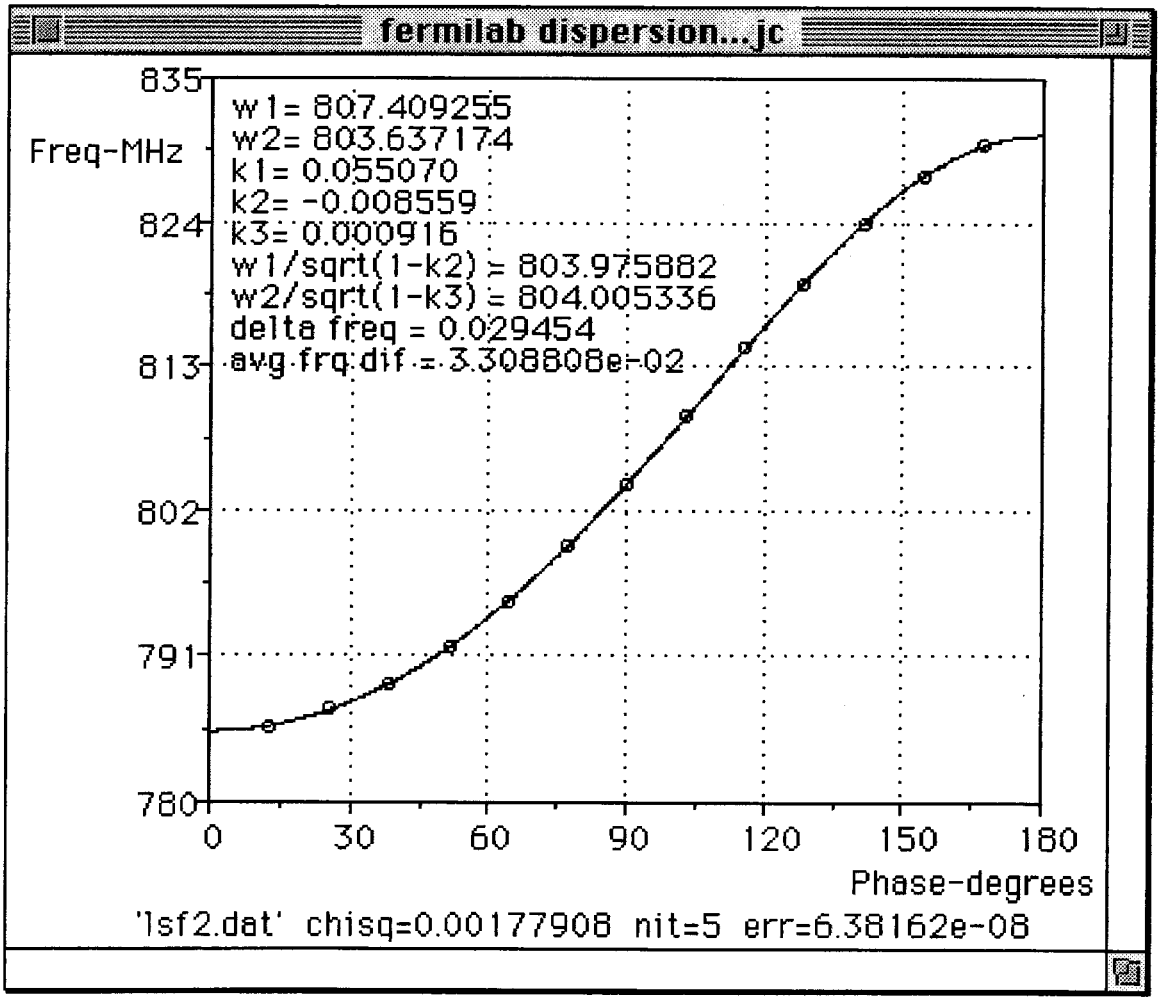


The file containing the frequency measurements must have the number of cells, or measurements, specified first and followed by the measured frequencies separated by any type of white space, i.e. carriage return, space, tab, etc.

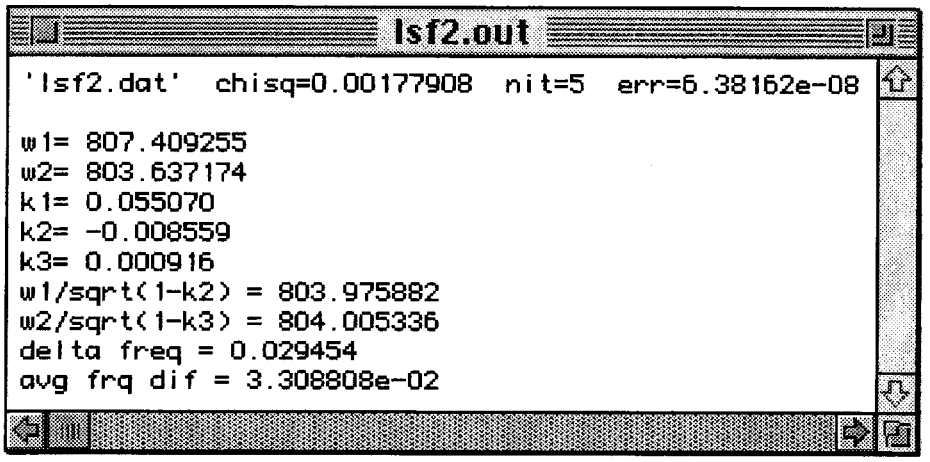


The first window displayed by the program keeps the user informed as the fit progresses. Each time the code returns from the Bevington gradient expansion algorithm the resulting chi square between the data and the equation 1 is displayed along with the relative change. When the change becomes less than 1 ppm the search is terminated and this window is replaced with a plot.





In addition to the plot a file is written to disk as shown below. The data file name is appended with the 4 characters '.out' and used for the file name.



References:

1) Nagel, D.E., Knapp,E.A., Knapp, B.C., "Coupled Resonator Model for Standing Wave Accelerator Tanks", Rev. Sci. Ins., Vol. 38, No. 11, pg. 1583, Nov. 1967.
2) Knapp, E.A., Knapp, B.C., Potter, J.M., "Standing Wave High Energy Linear Accelerator Structures", Rev. Sci. Ins., Vol. 39, No. 7, pg. 979, Jul. 1968.
3) Bevington, P.R., "Data Reduction and Error Analysis for the Physical Sciences", McGraw-Hill, 1969.
4) Press, W.H., Flannery, B.P., Teukolsky, S.A., Vetterling, W.T., "Numerical Recipes in C the Art of Scientific Computing", Cambridge, 1988.
5) Los Alamos.